LSU Communicating math

This presentation was created as part of the Communicating Math course (Spring 2025) within the graduate program at LSU. For this course, students are required to identify a mentor within the department (who may or may not be their PhD supervisor) and select an interesting topic for both a presentation and a written exposition.

Introduction to Dirichlet-to-Neumann Map

An example-oriented approach

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Louisiana State University

May 1,2025

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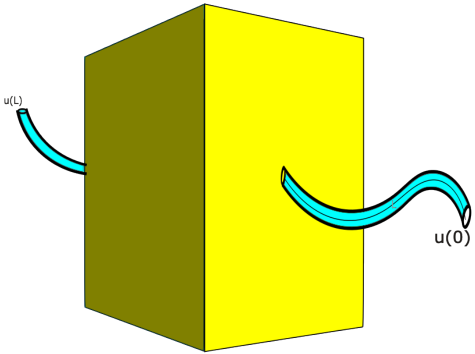
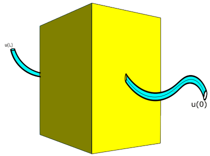
A “black-box”



A little bit of school physics (in 1D)

Let $\gamma(x)$ denote the conductivity of the wire('length' [0,L]) at x.



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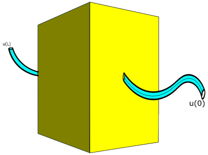

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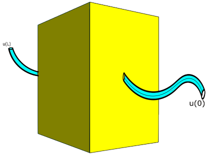

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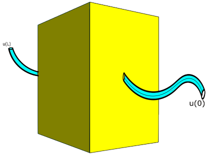

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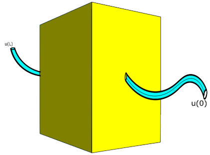

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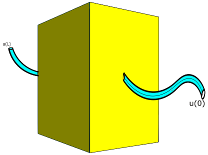

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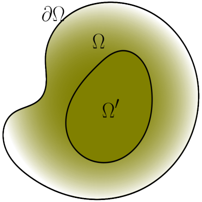
What can we measure directly?
(making $u(0) = 0$, $I = 1$)

$$\{u(0), u(L), \gamma(0)u'(0), \gamma(L)u'(L)\} = \{0, \int_0^L \frac{dy}{\gamma(y)}, 1, 1\}$$

In higher dimension

$\Omega' \subset \Omega \subset \mathbb{R}^n$.

$$\underbrace{i(x)}_{\text{not independent of } x} = -\gamma(x)\nabla u(x) \tag{2}$$



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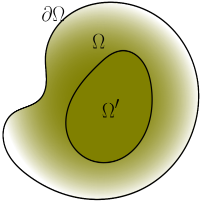

Steady state current:
$\int_{\partial\Omega'} i(x) \cdot \nu(x) dS(x) = 0$.

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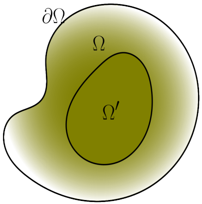

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$$\overset{IBP}{\Rightarrow} \int_{\Omega'} \nabla\cdot i(x)dx = 0 \;\; \forall\Omega' \subset \Omega$$

$$\Rightarrow \nabla\cdot i(x) = \nabla\cdot(\gamma(x)\nabla u(x)) = 0 \text{ in } \Omega$$

Prelude

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Voltage equation

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$\underbrace{\Lambda_{\gamma}(f)}_{\text{DtN map}} = g$ is a well defined map. The impedence Tomography problem is to infer data about γ from the DtN map.

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Sobolev Spaces

Definition ($H^k(\Omega)$ space)

$H^k(\Omega) = \{u : \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u \in L^2(\Omega)$ **in weak sense** with $\sum_{j=1}^{n} \alpha_j \leq k\}$

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- **The space $C^\infty(\bar{\Omega}) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$.**
- **For Ω to be a bounded domain with smooth boundary we can extend any $u \in H^1(\Omega)$ to its boundary, in fact on the whole $\mathbb{R}^n$.**

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$\mathrm{Tr}\, H^k(\Omega) := \{v \in L^2(\partial\Omega) : \exists u \in H^k(\Omega)\, \text{s.t.}\, Tr(u) = v\}$

Now the definition...

Consider the following Partial Differential equation:

Dirichlet Problem

$$\begin{aligned} \nabla \cdot (\gamma(x)\nabla u) &= 0 \,\text{on}\, \Omega \\ u &= f \,\text{on}\, \partial\Omega \end{aligned} \tag{4}$$

For $f \in \mathrm{Tr}H^2(\partial\Omega)$, we will put $\gamma(x) \in C^1(\bar{\Omega})$. **The dirichlet problem admits unique weak solution for such $\gamma(x)$ if $\gamma(x) > \epsilon > 0$.**

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In weak sense the range of Λ_γ is subset of the dual space $H^{\frac{1}{2}}(\partial\Omega)$.

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Example on Half-Space

The Laplacian: $\Delta u := \frac{\partial^2 u}{\partial x_1^2}$

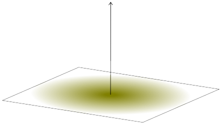
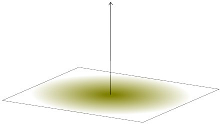


Figure: Upper half plane in $\mathbb{R}^3$. Note the boundary is actually $\mathbb{R}^2$

Example on Half-Space

The Laplacian: $\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$



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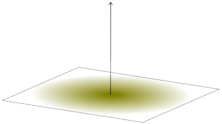

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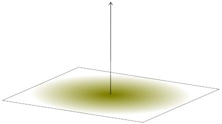

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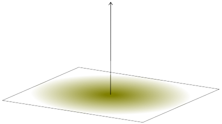
Denote the elements of $\mathbb{R}_+^{n+1}$, X as $(\mathbf{x}, y)$ such that $\mathbf{x} \in \mathbb{R}^n$, and $y \in (0, \infty)$.



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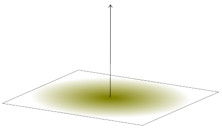

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The boiled down **Dirichlet problem** in (4):

$$\begin{aligned} \nabla \cdot (\nabla u(x)) = \Delta u = 0 \text{ in } \mathbb{R}_+^{n+1} \\ u = f \text{ in } \partial \mathbb{R}_+^{n+1} (= \mathbb{R}^n) \end{aligned} \tag{5}$$

We will assume f to be in certain space when it's required.

Example on Half-Space(Cont...)

Dirichlet Problem: $\Delta u(\mathbf{x}, y) = 0$ on $\mathbb{R}^{n+1}_+$; $u(\mathbf{x}, 0) = f(\mathbf{x})$ on $\mathbb{R}^n$.

The solution of this problem for any Ω:

$$u(X) = -\int_{\partial\Omega} f(\mathbf{z}) \frac{\partial G(X, \mathbf{z})}{\partial \nu} dS(\mathbf{z})$$

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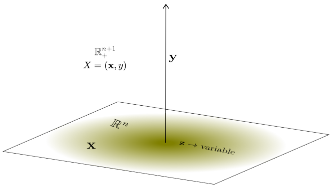
$$\therefore u(X) = \frac{2y}{c} \int_{\mathbb{R}^n} \frac{f(\mathbf{z}) d\mathbf{z}}{|X - \mathbf{z}|^{n+1}}; \; X \in \mathbb{R}^{n+1}_+$$



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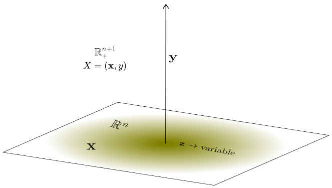

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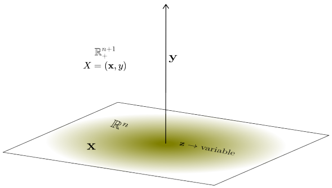

The expression is well defined for $f \in C^2(\mathbb{R}^n)$.

Thank You!

- L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.
- G. Guhlmann, *Inverse Problems: An Introduction*, IPAM Tutorial Notes, 2003.
- L. Borcea, "Electrical impedance tomography," *Inverse Problems*, vol. 18, no. 6, pp. R99–R136, 2002. "`https://public.websites.umich.edu/~borcea/Publications/EIT.pdf`" "doi:10.1088/0266-5611/18/6/201" (**For EIT**)